

# Diagonal multi-soliton matrix elements in finite volume

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We consider diagonal matrix elements of local operators between multi-soliton states in finite volume in the sine-Gordon model, and formulate a conjecture regarding their finite size dependence which is valid up to corrections exponential in the volume. This conjecture extends the results of Pozsgay and Takács which were only valid for diagonal scattering. In order to test the conjecture we implement a numerical renormalization group improved truncated conformal space approach. The numerical comparisons confirm the conjecture, which is expected to be valid for general integrable field theories. The conjectured formula can be used to evaluate finite temperature one-point and two-point functions using recently developed methods.

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## 1. INTRODUCTION

In this paper we continue a program to describe form factors in finite volume, initiated in [1, 2] and further developed and extended in the works [3–7]. Previously, finite volume form factors have also been studied in other approaches [8–10].

The present effort at describing finite volume form factors is directed towards an understanding of finite temperature correlation functions and other related cases where the spectral expansion of a quantity is ill-defined due to singularities associated with disconnected contributions present in operator matrix elements. A different attempt to find a spectral expansion for finite temperature correlators is Doyon's finite temperature form factor formalism [11].

We work in two-dimensional integrable quantum field theories, where the  $S$  matrix can be obtained from the bootstrap [12]. In such theories, using the scattering amplitudes as input it is possible to obtain a set of equations satisfied by the form factors [13]. The complete system of form factor equations, which provides the basis for the form factor bootstrap, was proposed in [14]. For a detailed and thorough exposition of the subject we refer to [15].

Finite volume form factors can be used as a tool to compute finite temperature correlation functions [2, 16–18], or matrix elements of local fields in a boundary setting [19]. Another application of the formalism is the extension of form factor perturbation theory, which was developed in [20] to describe non-integrable perturbations of integrable quantum field theories, to higher orders in [21].

One of the applications of our framework is that it allows for a direct comparison of solutions of the form factor axioms to field theory dynamics. In fact, this works the other way as well, since we can use well-established

results about the form factor solutions in a given theory to test our ideas or conjectures that arise in developing the finite volume form factor program. There is still place for developments, as we do not yet have a complete description of finite volume form factors for the case of non-diagonal scattering. The first steps were taken in [5] with a study of sine-Gordon breather and two-soliton form factors; later it was extended to multi-soliton states [7], using the framework for numerical evaluation of multi-soliton form factors developed in [22]. Another important direction is to incorporate exponential finite size corrections: so far this was only done for the so-called  $\mu$ -terms [4, 6], and even in that case the method is not yet entirely systematic.

Theories with non-diagonal scattering, in which the spectrum contains some nontrivial particle multiplets (typically organized into representations of some group symmetry), such as sine-Gordon or the  $O(3)$  nonlinear sigma model are very important for condensed matter applications (e.g. to spin chains or one-dimensional electron systems; for a review see [23]). The finite volume description of form factors can be used to develop a low-temperature and large-distance expansion for finite-temperature correlation functions, which could in turn be used to explain experimental data, e.g. from inelastic neutron scattering [16, 24]. Another interesting application is to extend the computation of one-point functions of bulk operators on a strip to the non-diagonal case. Finite volume methods are also a promising approach in the study of quantum quenches in integrable quantum field theories [19, 25].

In this paper we treat the sine-Gordon model as an example. It can be considered as the prototype of a non-diagonal scattering theory, and it has the advantage that its finite volume spectra and form factors can be studied very effectively numerically using the truncated conformal space approach (TCSA), originally developed

by Yurov and Zamolodchikov for the scaling Lee-Yang model [26], but later extended to the sine-Gordon theory [27]. Its exact form factors are also known in full generality [15, 28–31], and so it is a useful playground to test our theoretical ideas on finite volume form factors. However, before embarking on the present program, a technical problem had to be solved. Namely, it was clear from our earlier studies [5, 7] that truncated conformal space did not converge very well for diagonal matrix elements. In order to solve this problem we implemented the numerical renormalization group (NRG) improvement introduced by Konik and Adamov [32]. As we demonstrate, the resulting NRG-TCSA method proved to be accurate enough to perform a stringent test of our conjectures.

The paper is organized as follows. After a brief review of the necessary facts about sine-Gordon model and its finite volume soliton form factors in Section 2, we formulate our main conjecture in Section 3. In Section 4 we give a brief description of the numerical methods, and then present the results of our computations in Section 5. Section 6 is reserved for the conclusions.

## 2. SOLITONS IN FINITE VOLUME

### 2.1. Action and $S$ matrix

Sine-Gordon model is defined by the classical action

$$\mathcal{A} = \int d^2x \left( \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{m_0^2}{\beta^2} \cos \beta \Phi \right)$$

The spectrum of the quantum theory is generated by a doublet of a soliton and an antisoliton, both of mass  $M$ . Their exact  $S$  matrix can be written as [12]

$$S_{i_1 i_2}^{j_1 j_2}(\theta, \xi) = S_{i_1 i_2}^{j_1 j_2}(\theta, \xi) S_0(\theta, \xi) \quad (2.1)$$

$$\xi = \frac{\beta^2}{8\pi - \beta^2}$$

where the non-zero elements are

$$\begin{aligned} S_{++}^{++}(\theta, \xi) &= S_{--}^{--}(\theta, \xi) = 1 \\ S_{+-}^{+-}(\theta, \xi) &= S_{-+}^{+-}(\theta, \xi) = S_T(\theta, \xi) \\ S_{+-}^{+}(\theta, \xi) &= S_{-+}^{+}(\theta, \xi) = S_R(\theta, \xi) \end{aligned}$$

and

$$S_T(\theta, \xi) = \frac{\sinh\left(\frac{\theta}{\xi}\right)}{\sinh\left(\frac{i\pi - \theta}{\xi}\right)}, \quad S_R(\theta, \xi) = \frac{i \sin\left(\frac{\pi}{\xi}\right)}{\sinh\left(\frac{i\pi - \theta}{\xi}\right)}$$

$$S_0(\theta, \xi) = -\exp \left\{ -i \int_0^\infty \frac{dt}{t} \frac{\sinh \frac{\pi(1-\xi)t}{2}}{\sinh \frac{\pi \xi t}{2} \cosh \frac{\pi t}{2}} \sin \theta t \right\}$$

Besides the solitons, the spectrum of theory contains also breathers which are bound states of a soliton and antisoliton.

Sine-Gordon model can also be represented as a free massless boson conformal field theory (CFT) perturbed by a relevant operator, with the Hamiltonian

$$H = \int dx \frac{1}{2} : (\partial_t \Phi)^2 + (\partial_x \Phi)^2 : + \lambda \int dx : \cos \beta \Phi : \quad (2.2)$$

where the semicolon denotes normal ordering in terms of the modes of the  $\lambda = 0$  massless field. Due to anomalous dimension of the normal ordered cosine operator, the coupling constant  $\mu$  has dimension

$$\lambda \sim [\text{mass}]^{2-\beta^2/4\pi}$$

so it defines the mass scale of the model and the dimensionless coupling parameter is  $\beta$ .

### 2.2. Soliton form factors

The class of operators we consider consists of exponentials of the bosonic field  $\Phi$ . Their vacuum expectation value is known exactly [33]:

$$\mathcal{G}_a(\beta) = \langle e^{ia\beta\Phi} \rangle = \left[ \frac{M \sqrt{\pi} \Gamma\left(\frac{4\pi}{8\pi - \beta^2}\right)}{2\Gamma\left(\frac{\beta^2/2}{8\pi - \beta^2}\right)} \right]^{\frac{a^2 \beta^2}{4\pi}} \quad (2.3)$$

$$\times \exp \left\{ \int_0^\infty \frac{dt}{t} \left[ -\frac{a^2 \beta^2}{4\pi} e^{-2t} + \frac{\sinh^2\left(\frac{a}{4\pi} t\right)}{2 \sinh\left(\frac{\beta^2}{8\pi} t\right) \cosh\left(\left(1 - \frac{\beta^2}{8\pi}\right) t\right) \sinh t} \right] \right\}$$

with  $M$  denoting the soliton mass related to the coupling  $\lambda$  defined in via [34]

$$\lambda = \frac{2\Gamma(\Delta)}{\pi\Gamma(1-\Delta)} \left( \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2-2\Delta}\right) M}{2\Gamma\left(\frac{\Delta}{2-2\Delta}\right)} \right)^{2-2\Delta}, \quad \Delta = \frac{\beta^2}{8\pi} \quad (2.4)$$

Multi-soliton form factors, i.e.

$$F_{i_1 \dots i_N}^{\mathcal{O}}(\theta_1, \dots, \theta_N) = \langle 0 | \mathcal{O} | A_{i_N}(\theta_N) \dots A_{i_1}(\theta_1) \rangle, \quad (2.5)$$

( $A_{\pm}$  denoting a soliton/antisoliton and the ordering of rapidities  $\theta_1 > \dots > \theta_N$ ) have been constructed using several different approaches: the earliest construction is by Smirnov (reviewed in [15]), then a free field representation by Lukyanov [28, 29], and later in the work by Babujian et al. [30, 31]. Here we use formulae from Lukyanov's work [29] (in conjunction with its numerical evaluation method given in [22]); however, certain of his conventions are different and therefore we change the labeling of the form factors accordingly (see eqn. (2.10) below). The reason is that the form factors we use satisfy form factor bootstrap relations which are slightly different from Lukyanov's conventions; in this we conform to

the conventions of the papers [1, 2]. In our notations, the form factor equations are:

I. Lorentz-invariance

$$F_{i_1 \dots i_N}^{\mathcal{O}}(\theta_1 + \Lambda, \dots, \theta_N + \Lambda) = e^{s(\mathcal{O})\Lambda} F_{i_1 \dots i_N}^{\mathcal{O}}(\theta_1, \dots, \theta_N) \quad (2.6)$$

where  $s(\mathcal{O})$  is the Lorentz spin of the operator  $\mathcal{O}$ .

II. Exchange:

$$\begin{aligned} F_{i_1 \dots i_k i_{k+1} \dots i_N}^{\mathcal{O}}(\theta_1, \dots, \theta_k, \theta_{k+1}, \dots, \theta_N) = \\ S(\theta_k - \theta_{k+1}) \\ \times F_{i_1 \dots j_{k+1} j_k \dots i_N}^{\mathcal{O}}(\theta_1, \dots, \theta_{k+1}, \theta_k, \dots, \theta_N) \end{aligned} \quad (2.7)$$

III. Cyclic permutation:

$$F_{i_1 i_2 \dots i_N}^{\mathcal{O}}(\theta_1 + 2i\pi, \theta_2, \dots, \theta_N) = e^{2\pi i \omega(\mathcal{O})} F_{i_2 \dots i_N i_1}^{\mathcal{O}}(\theta_2, \dots, \theta_N, \theta_1) \quad (2.8)$$

where  $\omega(\mathcal{O})$  is the mutual locality index between the operator  $\mathcal{O}$  and the asymptotic field that creates the solitons.

IV. Kinematical singularity

$$\begin{aligned} -i \operatorname{Res}_{\theta=\theta'} F_{i k i_1 \dots i_N}^{\mathcal{O}}(\theta + i\pi, \theta', \theta_1, \dots, \theta_n) = \\ C_{ik'} \left( \delta_k^{k'} - e^{2\pi i \omega(\mathcal{O})} S_{k i_1}^{k_1 j_1}(\theta' - \theta_1) S_{k_1 i_2}^{k_2 j_2}(\theta' - \theta_2) \right. \\ \left. \dots S_{k_{n-1} i_n}^{k' j_n}(\theta' - \theta_n) \right) F_{j_1 \dots j_N}^{\mathcal{O}}(\theta_1, \dots, \theta_N) \end{aligned} \quad (2.9)$$

where  $C$  is the charge conjugation matrix.

There is a further equation that relates form factors containing breathers to those containing only solitons, but it is not needed for multi-soliton states. These equations are supplemented by the assumption of maximum analyticity (i.e. that the form factors are meromorphic functions which only have the singularities prescribed by the axioms) and possible further conditions expressing properties of the particular operator whose form factors are sought.

The form factors of the operator

$$\mathcal{O}_a = e^{ia\beta\Phi}$$

which satisfy equations (2.6-2.9) with the locality index

$$\omega(\mathcal{O}_a) = a \bmod 1$$

can be obtained from

$$\begin{aligned} F_{\sigma_1 \dots \sigma_{2n}}^a(\theta_1, \dots, \theta_{2n}) &= (-1)^n \mathcal{F}_{-\sigma_{2N} \dots -\sigma_1}^{(a)}(\theta_{2n}, \dots, \theta_1) \\ &= (-1)^n \mathcal{F}_{\sigma_{2n} \dots \sigma_1}^{(-a)}(\theta_{2n}, \dots, \theta_1) \end{aligned} \quad (2.10)$$

where the functions  $\mathcal{F}$  (originally derived by Lukyanov in [29]) are specified in appendix A of [7].

### 2.3. Soliton form factors in finite volume

The formulae for finite volume form factors, derived in [1, 2], were generalized for the case of non-diagonal theories in [5] and further investigated in [7]. Here we only recall the necessary facts; for more details the reader is referred to the original papers.

In finite volume  $L$ , the space of multi-soliton states can be labeled by momentum quantum numbers  $I_1, \dots, I_N$ . We introduce the following notation for them:

$$|\{I_1, I_2, \dots, I_N\}\rangle_L^{(r)} \quad (2.11)$$

where the index  $r$  enumerates the eigenvectors of the  $n$ -soliton transfer matrix, which can be written as

$$\begin{aligned} \mathcal{T}(\vartheta | \{\theta_1, \dots, \theta_N\})_{i_1 \dots i_N}^{j_1 \dots j_N} = \\ \mathcal{S}_{a i_1}^{c_1 j_1}(\vartheta - \theta_1) \mathcal{S}_{c_1 i_2}^{c_2 j_2}(\vartheta - \theta_2) \dots \mathcal{S}_{c_{N-1} i_N}^{a j_N}(\vartheta - \theta_N) \end{aligned}$$

where  $\theta_1, \dots, \theta_N$  are particle rapidities. The transfer matrix can be diagonalized simultaneously for all values of  $\vartheta$ :

$$\begin{aligned} \mathcal{T}(\vartheta | \{\theta_1, \dots, \theta_N\})_{i_1 \dots i_N}^{j_1 \dots j_N} \Psi_{j_1 \dots j_N}^{(r)}(\{\theta_k\}) = \\ t^{(r)}(\vartheta, \{\theta_k\}) \Psi_{i_1 \dots i_N}^{(r)}(\{\theta_k\}) \end{aligned}$$

We can assume that the wave function amplitudes  $\Psi^{(r)}$  are normalized and form a complete basis:

$$\begin{aligned} \sum_{i_1 \dots i_N} \Psi_{i_1 \dots i_N}^{(r)}(\{\theta_k\}) \Psi_{i_1 \dots i_N}^{(s)}(\{\theta_k\})^* &= \delta_{rs} \quad (2.12) \\ \sum_r \Psi_{i_1 \dots i_N}^{(r)}(\{\theta_k\}) \Psi_{j_1 \dots j_N}^{(r)}(\{\theta_k\})^* &= \delta_{i_1 j_1} \dots \delta_{i_N j_N} \end{aligned}$$

these eigenfunctions describe the possible polarizations of the  $N$  particle state with rapidities  $\theta_1, \dots, \theta_N$  inside the  $2^N$  dimensional internal space indexed by  $i_1 \dots i_N$ . The transfer matrix can be diagonalized using the algebraic Bethe Ansatz (cf. Appendix A of [5]), which enables one to compute the exact form of eigenvalues  $t^{(r)}$  and eigenvectors  $\Psi^{(r)}$ .

The rapidities of the particles in the state (2.11) can be determined by solving the quantization conditions

$$\begin{aligned} Q_j^{(r)}(\theta_1, \dots, \theta_n) = \\ ML \sinh \theta_j + \delta_j^{(r)}(\theta_1, \dots, \theta_N) = 2\pi I_j, \quad k = 1, \dots, N \\ \delta_j^{(r)}(\theta_1, \dots, \theta_N) = -i \log t^{(r)}(\theta_j, \{\theta_k\}) \end{aligned} \quad (2.13)$$

When considering rapidities which solve these equations with given quantum numbers  $I_1, \dots, I_N$  and a specific polarization state  $r$ , they will be written with a tilde as  $\tilde{\theta}_1, \dots, \tilde{\theta}_N$ .

Using the above ingredients, the finite volume matrix

elements can then be written as [5, 7]

$$\left| {}^{(s)}\langle \{I'_1, \dots, I'_M\} | \mathcal{O}(0, 0) | \{I_1, \dots, I_N\} \rangle_L^{(r)} \right| = \left| \frac{F^{\mathcal{O}(s)}(\tilde{\theta}'_M, \dots, \tilde{\theta}'_1 | \tilde{\theta}_1, \dots, \tilde{\theta}_N)^{(r)}}{\sqrt{\rho^{(r)}(\tilde{\theta}_1, \dots, \tilde{\theta}_N) \rho^{(s)}(\tilde{\theta}'_1, \dots, \tilde{\theta}'_M)}} \right| + O(e^{-\mu L}) \quad (2.14)$$

where  $\rho^{(r)}$  and  $\rho^{(s)}$  denote the density of states of types  $r$  and  $s$ , which can be calculated as the Jacobi determinant of the Bethe-Yang equations (2.13), considering them as a mapping from the rapidity to quantum number space:

$$\rho^{(r)}(\theta_1, \dots, \theta_N) = \det \left\{ \frac{\partial Q_j^{(r)}}{\partial \theta_k} \right\}_{j,k=1, \dots, N} \quad (2.15)$$

Furthermore,

$$\begin{aligned} F^{\mathcal{O}(s)}(\theta'_M, \dots, \theta'_1 | \theta_1, \dots, \theta_N)^{(r)} = \\ \sum_{j_1 \dots j_M} \sum_{i_1 \dots i_N} \Psi_{j_1 \dots j_M}^{(s)}(\{\theta'_k\})^* \\ \times F_{j_M \dots j_1 i_1 \dots i_N}^{\mathcal{O}}(\theta'_M + i\pi, \dots, \theta'_1 + i\pi, \theta_1, \dots, \theta_N) \\ \times \Psi_{i_1 \dots i_N}^{(r)}(\{\theta_k\}) \end{aligned} \quad (2.16)$$

is the  $(s, r)$ -polarized form factor (the bar denotes the antiparticle). The absolute value in (2.14) is necessary to account for the different phase conventions of the multi-particle states used in the form factor bootstrap and in the finite volume calculations.

### 3. A CONJECTURE FOR DIAGONAL MATRIX ELEMENTS

Relation (2.14) is only valid for matrix elements with no disconnected pieces, i.e. when the rapidities in the two finite volume states are all different from each other. If there are particles with exactly coinciding rapidities in the two states, i.e.  $\theta'_k = \theta_l$  for some  $k$  and  $l$ , then there are further contributions. Note that equality of two quantum numbers such as  $I'_k = I_l$  is not sufficient for the presence of a disconnected contribution, as the corresponding rapidities will in general be different due to the terms involving the phase shifts  $\delta_j^{(r)}$ . Therefore such terms are only present for the case when the two sets of quantum numbers are exactly identical, and also in the special case when the two states each contain a particle with exactly zero rapidity. At present, the disconnected terms are only known for states with diagonal scattering; the form of these contributions was obtained in [2].

Here we present a conjecture for diagonal matrix elements with non-diagonal scattering, which is an educated guess based on the results valid for diagonal scattering and also on some lessons learned from nested Bethe Ansatz systems [35, 36]. For diagonal scattering, the for-

mula introduced in [2] states that the diagonal finite volume matrix element can be computed as

$$\begin{aligned} \langle \{I_1, \dots, I_N\} | \mathcal{O}(0, 0) | \{I_1, \dots, I_N\} \rangle_L = \\ \frac{1}{\rho(\{1, \dots, N\})_L} \sum_{A \subset \{1, 2, \dots, N\}} \mathcal{F}(A)_L \rho(\bar{A})_L + O(e^{-\mu L}) \end{aligned} \quad (3.1)$$

where  $\bar{A} = \{1, 2, \dots, N\} \setminus A$ . Let us denote the elements of the sets  $A$  and  $\bar{A}$  as

$$\begin{aligned} A &= \{A_1, \dots, A_l\} \\ \bar{A} &= \{\bar{A}_1, \dots, \bar{A}_{N-l}\} \end{aligned}$$

with  $l = |A|$  the cardinal number (number of elements) of the set  $A$ . Then

$$\rho(\bar{A})_L = \rho(\tilde{\theta}_{\bar{A}_1}, \dots, \tilde{\theta}_{\bar{A}_{N-l}})_L$$

is the  $l$ -particle Bethe-Yang Jacobi determinant involving only the subset  $\bar{A}$  of the  $N$  particles, and

$$\mathcal{F}(A)_L = F_l^s(\tilde{\theta}_{A_1}, \dots, \tilde{\theta}_{A_l})$$

where

$$\begin{aligned} F_l^s(\theta_1, \dots, \theta_l)_{i_1 \dots i_l} = \\ \lim_{\epsilon \rightarrow 0} F^{\mathcal{O}}(\theta_l + i\pi + \epsilon, \dots, \theta_1 + i\pi + \epsilon, \theta_1, \dots, \theta_l)_{\bar{i}_l \dots \bar{i}_1 i_1 \dots i_l} \end{aligned}$$

is the so-called symmetric evaluation of the diagonal form factor involving the particles in set  $A$  (the bar denotes the antiparticle).

The main observation is that the density  $\rho$  is only well-defined for states that have an internal state which is an eigenvector of the multi-soliton transfer matrix. Therefore, in order to define the disconnected term corresponding to a given subset  $A$ , the  $N$ -soliton wave function amplitude must be decomposed accordingly. Let us suppose that

$$|\{I_1, I_2, \dots, I_N\}\rangle_L^{(r)}$$

is a finite volume state which corresponds to a wave function eigenvector

$$\Psi_{i_1 \dots i_N}^{(r)}(\{\theta_k\})$$

Using the orthogonality and completeness of the  $\Psi$  amplitudes (eqn. (2.12)), for any subset  $A \subset \{1, 2, \dots, N\}$  we can define appropriate branching coefficients  $\mathcal{C}$  to split the wave function into pieces given by tensor products of transfer matrix eigenvectors for the sets  $A$  and  $\bar{A}$  as follows:

$$\begin{aligned} \Psi_{i_1 \dots i_N}^{(r)}(\{\theta_k\}) = \sum_{s,t} \mathcal{C}_{st}^{(r)}(\{\theta_k\} | A) \\ \times \Psi_{\{i_k\}_{k \in A}}^{(s)}(\theta_{A_1}, \dots, \theta_{A_l}) \Psi_{\{i_k\}_{k \in \bar{A}}}^{(t)}(\{\theta_k\}_{k \in \bar{A}}) \end{aligned}$$

where the sum over  $s$  runs over all possible polarization states (transfer matrix eigenvectors) for  $l$  particles, and the sum over  $t$  runs similarly for polarization states of  $N-l$  particles. From (2.12) it follows that the branching coefficients are normalized as

$$\sum_{s,t} \left| \mathcal{C}_{st}^{(r)}(\{\theta_k\} | A) \right|^2 = 1$$

The conjectured generalization of (3.1) is then

$$\begin{aligned} {}^{(r)}\langle \{I_1, \dots, I_N\} | \mathcal{O}(0,0) | \{I_1, \dots, I_N\} \rangle_L^{(r)} = \\ \frac{1}{\rho^{(r)}(\{1, \dots, N\})_L} \sum_{A \subset \{1, 2, \dots, N\}} \quad (3.2) \\ \sum_{s,t} \left| \mathcal{C}_{st}^{(r)}(\{\tilde{\theta}_k\} | A) \right|^2 \mathcal{F}^{(s)}(A)_L \rho^{(t)}(\bar{A})_L + O(e^{-\mu L}) \end{aligned}$$

where

$$\mathcal{F}^{(s)}(A)_L = \lim_{\epsilon \rightarrow 0} F^{\mathcal{O}(s)}(\tilde{\theta}_{A_l} + \epsilon, \dots, \tilde{\theta}_{A_1} + \epsilon | \tilde{\theta}_{A_1}, \dots, \tilde{\theta}_{A_l})^{(s)}$$

is the symmetric diagonal limit of the  $(s, s)$ -polarized form factor from (2.16) and

$$\rho^{(t)}(\bar{A})_L = \rho^{(t)}(\tilde{\theta}_{\bar{A}_1}, \dots, \tilde{\theta}_{\bar{A}_{N-l}})$$

is the density of states in the  $\Psi^{(t)}$  channel.

It is easy to see that eqn. (3.2) reproduces (3.1) for diagonal scattering, as in that case there is only a single polarization state for any given number of particles and the branching coefficients  $\mathcal{C}$  are all equal to 1. Indeed the structure of (3.2) is very obvious, the only ambiguity is which evaluation of the diagonal form factor to use; for that we substituted the symmetric one, to keep the correspondence with (3.1).

The first nontrivial case arises when  $N = 2$ . In this subspace there are two states with non-diagonal scattering, which contain a soliton and an antisoliton. The wave function amplitudes of the transfer matrix eigenstates are

$$\Psi^{(+)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \Psi^{(-)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

which are nothing else but the branching coefficients of these states in a (one-soliton)  $\times$  (one-soliton) basis; it is a great simplifying feature that these are rapidity independent. The general formula (3.2) specified to this case yields

$$\begin{aligned} {}^{(\pm)}\langle \{I_1, I_2\} | \mathcal{O}(0,0) | \{I_1, I_2\} \rangle_L^{(\pm)} = \\ \frac{1}{\rho^{(\pm)}(\tilde{\theta}_1, \tilde{\theta}_2)} \left[ \lim_{\epsilon \rightarrow 0} F_4^{\mathcal{O}(\pm)}(\tilde{\theta}_2 + \epsilon, \tilde{\theta}_1 + \epsilon | \tilde{\theta}_1, \tilde{\theta}_2)^{(\pm)} + \right. \\ \left. F_{2s}^{\mathcal{O}}(\rho_1(\tilde{\theta}_1) + \rho_1(\tilde{\theta}_2)) + \rho^{(\pm)}(\tilde{\theta}_1, \tilde{\theta}_2) \langle \mathcal{O} \rangle \right] \quad (3.3) \end{aligned}$$

where

$$\rho_1(\theta) = ML \cosh \theta$$

is the one-soliton state density, the diagonal one-soliton form factor is

$$F_{2s}^{\mathcal{O}} = F_{+-}^{\mathcal{O}}(\theta + i\pi, \theta) = F_{-+}^{\mathcal{O}}(\theta + i\pi, \theta)$$

and it is independent of  $\theta$  due to Lorentz invariance. In addition, the  $(\pm, \pm)$ -polarized two-soliton-two-soliton form factors are given by

$$\begin{aligned} F_4^{\mathcal{O}(s)}(\theta'_2, \theta'_1 | \theta_1, \theta_2)^{(r)} = \\ \frac{1}{2} \left[ F_{+-+}^{\mathcal{O}}(\theta'_2 + i\pi, \theta'_1 + i\pi, \theta_1, \theta_2) \right. \\ + r F_{+--+}^{\mathcal{O}}(\theta'_2 + i\pi, \theta'_1 + i\pi, \theta_1, \theta_2) \\ + s F_{-++-}^{\mathcal{O}}(\theta'_2 + i\pi, \theta'_1 + i\pi, \theta_1, \theta_2) \\ \left. + rs F_{-+-+}^{\mathcal{O}}(\theta'_2 + i\pi, \theta'_1 + i\pi, \theta_1, \theta_2) \right] \end{aligned}$$

with  $r, s = \pm 1$ . The rapidities  $\tilde{\theta}_{1,2}$  can be obtained by solving the quantization conditions:

$$\begin{aligned} Q_1^{(\pm)}(\theta_1, \theta_2) &= ML \sinh \theta_1 + \delta_{\pm}(\theta_1 - \theta_2) = 2\pi I_1 \\ Q_2^{(\pm)}(\theta_1, \theta_2) &= ML \sinh \theta_2 + \delta_{\pm}(\theta_2 - \theta_1) = 2\pi I_2 \quad (3.4) \end{aligned}$$

where the phase-shifts  $\delta_{\pm}$  are defined from the eigenvalues of the two-soliton  $S$ -matrix in the neutral subspace by

$$\begin{aligned} \mathcal{S}_+(\theta) &= \mathcal{S}_{+-}^{+-}(\theta) + \mathcal{S}_{-+}^{-+}(\theta) = -e^{i\delta_+(\theta)} \\ \mathcal{S}_-(\theta) &= \mathcal{S}_{+-}^{+-}(\theta) - \mathcal{S}_{-+}^{-+}(\theta) = e^{i\delta_-(\theta)} \end{aligned}$$

where the minus sign introduced in the first line is a redefinition, which ensures that the phase-shifts are odd and continuous functions of the rapidity  $\theta$ . Due to this convention the even states are quantized with half-integer, while the odd states are quantized with integer quantum numbers. The densities  $\rho^{(\pm)}$  of neutral two-soliton states can be written as the Jacobi determinant [5]

$$\rho^{(\pm)}(\theta_1, \theta_2) = \begin{vmatrix} \frac{\partial Q_1^{(\pm)}}{\partial \theta_1} & \frac{\partial Q_1^{(\pm)}}{\partial \theta_2} \\ \frac{\partial Q_2^{(\pm)}}{\partial \theta_1} & \frac{\partial Q_2^{(\pm)}}{\partial \theta_2} \end{vmatrix}$$

## 4. NUMERICAL METHODS

### 4.1. TCSA in a nutshell

To evaluate the form factors numerically, we use the truncated conformal space approach (TCSA) pioneered by Yurov and Zamolodchikov [26]. The basic idea of this approach is to consider the field theoretic Hamiltonian



on the space of states of the conformal field theory, and truncate the basis of this space to finitely many vectors by placing an appropriate upper energy cutoff. The extension to the sine-Gordon model was developed in [27] and has found numerous applications since then. The Hilbert space can be split by the eigenvalues of the topological charge  $\mathcal{Q}$  (or winding number) and the spatial momentum  $P$ , where the eigenvalues of the latter are of the form

$$\frac{2\pi s}{L}$$

$s$  is called the ‘conformal spin’. In sectors with vanishing topological charge, we can reduce the size of the Hilbert space using the symmetry of the Hamiltonian under conjugation of the solitonic charge:

$$\mathcal{C} : \quad \Phi(x, t) \rightarrow -\Phi(x, t)$$

To fix our notations, we give here the formal definition for the cut-off Hilbert space:

$$\begin{aligned} \mathcal{H}_{s,q}^{(\pm)}(e_{cut}) = \\ \{ |\Psi\rangle : \mathcal{Q}|\Psi\rangle = q|\Psi\rangle, (L_0 - \bar{L}_0)|\Psi\rangle = s|\Psi\rangle, \mathcal{C}|\Psi\rangle = \pm|\Psi\rangle \\ \text{and } (L_0 + \bar{L}_0 - 1/12)|\Psi\rangle = e|\Psi\rangle \text{ with } e \leq e_{cut} \} \end{aligned}$$

where  $|\Psi\rangle$  runs over the Hilbert space of the ultraviolet massless boson with quasi-periodic boundary conditions

$$\Phi(x + L, t) = \Phi(x, t) + \frac{2\pi}{\beta} q \quad q \in \mathbb{Z}$$

$L_0$  and  $\bar{L}_0$  are the usual Virasoro generators, and  $e_{cut}$  is the (dimensionless) cut-off parameter. The upper index  $\pm$  corresponding to projections under  $\mathcal{C}$  is applicable only in the  $\mathcal{Q} = 0$  sectors.

For the purpose of form factor calculations, we choose the operator

$$\mathcal{O} =: e^{i\beta\Phi} :$$

where the semicolons denote normal ordering with respect to the  $\lambda = 0$  free massless boson modes. It has the conformal dimensions

$$\Delta_{\mathcal{O}} = \bar{\Delta}_{\mathcal{O}} = \frac{\beta^2}{8\pi}$$

Using relation (2.4) we can express all energy levels and matrix elements in units of (appropriate powers of) the soliton mass  $M$ , and we also introduce the dimensionless volume variable  $l = ML$ . The general procedure is the same as in [1, 2]: the particle content of energy levels can be identified by matching the numerical TCSA spectrum against the predictions of the Bethe-Yang equations (2.13). After identification, one can compare the appropriate matrix elements to the theoretical values given by (2.14) and (3.2). There are some technical issues in the identification of states due to level crossings, which can also affect numerical accuracy; for a discussion of these we refer to [3, 5].

## 4.2. The numerical renormalization group in TCSA

As mentioned in the introduction, our earlier studies showed [5, 7] that TCSA did not converge very well for diagonal matrix elements. This means that the numerically evaluated matrix elements showed a marked dependence on the cutoff  $e_{cut}$ . It is also known from earlier experience that TCSA converges faster for smaller values of  $\beta$  (or  $\xi$ ), and this was also clear from previous calculations [5, 7, 21, 37]: decreasing  $\xi$  the cut-off dependence was reduced and the agreement between the numerical TCSA results and the predictions of the finite volume form factor formalism was improved at the same time.

Following the proposal of Konik and Adamov [32], we use a Wilsonian type numerical renormalization group (NRG) to improve the precision of TCSA. This allows us to use much higher values for the cutoff than for the usual TCSA. Usually, the attainable cutoff is in the range  $e_{cut} = 20 \dots 26$  (depending on the sector and the value of  $\beta$ ), which corresponds to an upper limit of around 20000 states. E.g. in the  $\mathcal{Q} = 2, s = 0$  sector at  $\xi = \frac{2}{7}$ ,  $e_{cut} = 26$  translates to 23771 states. With the NRG improvement, it is possible to take into account the effect of several hundred thousand states, e.g. in the previous example we could take into account 840000 states corresponding to the cutoff value of 38 (the number of states increases exponentially with the cutoff).

Our implementation of the NRG in TCSA is as follows. Let us take a given value  $e_{cut}$  of the cutoff with  $N$  states, which we order by increasing value of their conformal energy eigenvalue. We start with the lowest lying  $N_0$  of them (which is chosen to correspond to some lower value  $e_{cut,0}$  of the cutoff). We split the remaining states into  $k$  ‘shells’, each containing  $N_1$  states and a ‘remainder shell’ of  $N_r$  states such that

$$N = N_0 + kN_1 + N_r$$

The idea is that first we diagonalize the lowest  $N_0 \times N_0$  block of the Hamiltonian and retain the first  $n$  lowest lying eigenvalues with their corresponding eigenvectors. Then we add the first shell to the basis and recompute the lowest  $n$  eigenvalues and corresponding eigenvectors, which are now  $N_0 + N_1$  dimensional. In the next step, we add to these modified eigenvectors the next shell, and recompute the eigenvalues and eigenvectors, which in the original conformal basis will now have  $N_0 + 2N_1$  dimensions. Repeating the procedure  $k$  times and applying it to the remainder shell, we arrive at results for the first  $n$  eigenvectors in the  $N$  dimensional Hilbert space.

It is obvious that this procedure introduces a new element of approximation, namely we partially neglect part the matrix elements mixing the higher shells (part of it is retained as the starting  $N_0$  basis vectors at step  $l$  will contain components from the previous  $l - 1$  shells). In terms of the parameters there are two sources of error: setting i)  $k > 1$  and ii)  $n \neq N_0$ . E.g. for  $n = N_0$  the procedure would be exact when  $k = 1$ , i.e. if the completion is done in one step. The error resulting from these

sources is difficult to control. Nevertheless, our experience shows that by choosing the starting dimension  $N_0$  and the number of steps  $k$  well, the results are very much improved. For low values of the total dimension  $N$ , a direct comparison to the exact spectrum in the truncated space is possible and this was carried out as a preliminary study to assess the reliability of the method (see later).

In principle, once the cutoff is high enough one can apply a perturbative renormalization group to extrapolate to infinite cutoff [32, 38–40]. This predicts that for a quantity  $\mathcal{M}$  with a scaling dimension  $y$ , the cutoff dependence would be

$$\mathcal{M}(e_{cut}) = \mathcal{M}_\infty + A(e_{cut})^{-y} + \dots \quad (4.1)$$

Our numerical data shows that this scaling region has not yet been reached for the diagonal form factors using up to approximately 500000 states ( $e_{cut} = 36$ ). However, we observed that the more naive method of using the first order formula (4.1) and letting  $y$  to be a free fitting parameter works very well, and we always obtained an exponent  $y$  close to 2.

This is explained by the following considerations. A simple guess for the next correction is

$$\mathcal{M}(e_{cut}) = \mathcal{M}_\infty + A(e_{cut})^{-y} + B(e_{cut})^{-y-(2-2\Delta)} + \dots \quad (4.2)$$

since the coupling has a dimension of  $[\text{energy}]^{2-2\Delta}$ . Indeed, this is found to work surprisingly well. For the operator whose matrix elements are considered here one has  $y = 2\Delta$ , so that (4.2) reduces to

$$\mathcal{M}(e_{cut}) = \mathcal{M}_\infty + A(e_{cut})^{-2\Delta} + B(e_{cut})^{-2} + \dots$$

and it turns out that in all of our calculations the cut-off dependence is dominated by the second exponent. In some cases, keeping the lower order term produces a false numerical fit, so sometimes its omission necessary in order to get a meaningful extrapolation.

In Fig. 4.1 numerical evaluations of the vacuum expectation value at  $\xi = \frac{2}{7}$  are displayed. The NRG flow of  $|\langle O \rangle|$  was initiated at  $e_{cut,0} = 20$  (5614 states). For  $e_{cut} = 23$  (16032 states) TCSA calculations with and without NRG were also carried out. The difference between the two sets of data points is small. Note that using a step size of 100 states produced data differing from the displayed NRG-TCSA points only in the 6th digits. Results for  $e_{cut} = 31$  (181046 states) and two-exponent extrapolation (4.2) using data up to  $e_{cut} = 31$  are also shown.

In conclusion, utilizing the NRG (and RG) improvement it is possible to obtain a satisfactory agreement between the theoretical and TCSA vacuum expectation values. Based on this preliminary study, it is also reasonable to believe that the NRG error remains small for other diagonal matrix elements, thus the NRG improved TCSA should be reliable.

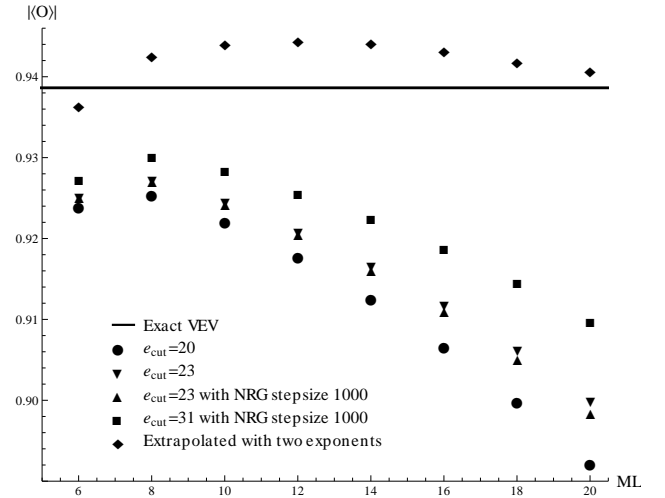


Figure 4.1: TCSA evaluation of the vacuum expectation value at  $\xi = 2/7$ , with and without NRG, compared to the exact result (2.3). An extrapolation using (4.2) is also shown.

## 5. RESULTS

### 5.1. Diagonal scattering

Before presenting the more interesting results corroborating our conjecture (3.2) we show some data confirming (3.1), which also serve to test the accuracy of the NRG-TCSA method.

This case was already investigated in [7]. However, in that work we found that the agreement between theory and TCSA results was rather unsatisfactory, especially for  $\xi = 2/7$ . In contrast, the NRG greatly improves the situation, as shown in Fig. 5.1 where two and three soliton diagonal matrix elements corresponding to the lowest lying states in the appropriate sectors are plotted. The data are parametrized by the compactification radius of the bosonic field defined by

$$R = \frac{\sqrt{4\pi}}{\beta}$$

and related to  $\xi$  as

$$\xi = \frac{1}{2R^2 - 1}$$

$\xi = 2/7$  and  $\xi = 50/311$  correspond to  $R = 1.5$  and  $1.9$ , respectively. For the two soliton data we employed a second order extrapolation (4.2) using NRG-TCSA data up to  $e_{cut} = 34$  (corresponding to approx. 277 thousand states). For the three soliton data an extrapolation was used omitting the  $e_{cut}^{-2\Delta}$  term and NRG-TCSA data up to  $e_{cut} = 37$  (corresponding to approx. 140 thousand states). The agreement is quite good even in the less trivial three-soliton case, where six soliton form factors are encountered on the bootstrap side.

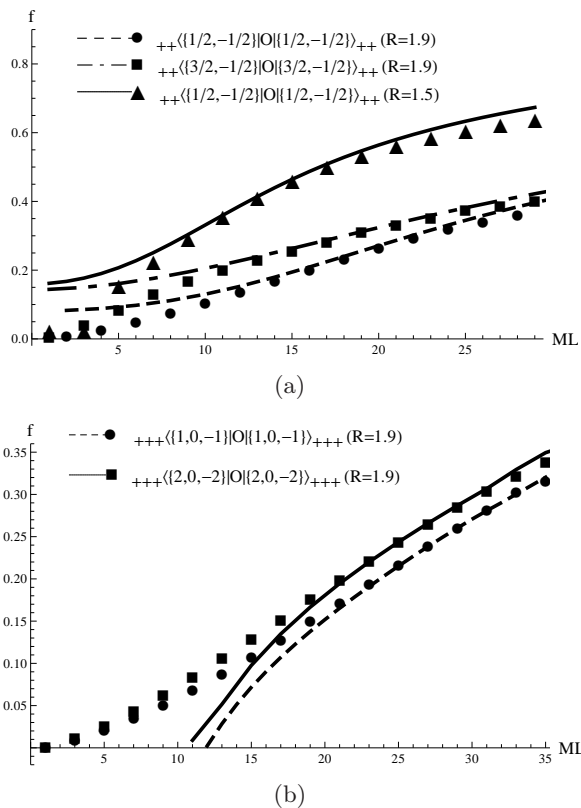


Figure 5.1: Diagonal matrix elements with (a) 2 solitons and (b) 3 solitons, all of them of the same (+) charge. (Here and in the subsequent figures  $f$  denotes the matrix element values in units of  $M$ .)

## 5.2. Non-diagonal scattering

Now we turn to the case of non-diagonal scattering. In Fig. 5.2 a large number of comparative data is shown for diagonal matrix elements in the  $Q = 0$  sector and compactification radii  $R = 1.5, 1.7$  and  $1.9$  (corresponding to  $\xi = 2/7, 50/239$  and  $50/311$ , respectively). All the NRG-TCSA data are improved by second order RG extrapolation (4.2) using measurements up to  $e_{cut} = 31$  (approx. 180 thousand states). The agreement is again very convincing. Apart from the consistently small systematic errors there are a few outlier points due to level crossings (for a discussion see [7]), and also the breakdown of the extrapolation in larger volume is seen. The latter could again be remedied by omitting the  $e_{cut}^{-2\Delta}$  term from (4.2).

## 6. CONCLUSIONS

In this paper we discussed diagonal matrix elements between multi-soliton states in sine-Gordon theory. Because of the strong cutoff dependence of diagonal matrix elements observed in previous works [5, 7], we first had to improve the TCSA method by taking into account

more states from the conformal Hilbert space. This was accomplished by implementing the numerical renormalization group method proposed by Konik and Adamov [32], and then improving the resulting NRG-TCSA data further by cutoff extrapolation, using the TCSA renormalization group introduced in [32, 38].

The theoretical description for diagonal matrix elements in states with diagonal scattering was known from [2], and so we could use expectation values computed in two-soliton states and three-soliton states, with all solitons having positive topological charge, to verify the accuracy of the method. In this way we demonstrated conclusively that the discrepancies observed previously [7] were really due to cutoff effects. In fact, in our previous work we did not even attempt to show data for diagonal matrix elements with  $Q = +3$  three-soliton states because of the large inaccuracy, but the NRG-TCSA method allowed for a satisfactory comparison of these as well.

The main theoretical result of this paper is a conjecture for diagonal matrix elements in states with non-diagonal scattering, formulated in Section 3. For the already known case of diagonal scattering, this conjecture reduces to the well established formula (3.1), first obtained in [2]. Therefore we verified the first nontrivial case, which was diagonal matrix elements in  $Q = 0$  two-soliton states, i.e. states containing a soliton and an antisoliton. The conjecture did fit the data in a very convincing way.

It would be interesting to provide further verification with  $Q = +1$  three-soliton states; we tried to identify such states in the spectrum, but due to the density of the spectrum and the exponential finite size corrections for small volume, this proved elusive so far. There are certain possibilities to improve this situation: using the exact NLIE description [41–45] of the finite volume levels instead of the approximation provided by the Bethe-Yang equations (2.13), and implementing further improvements to the TCSA method. However, this is outside the scope of this work, and we hope to return to this problem at some point in the future.

Nevertheless, even by restricting ourselves to the results for which we have sufficient numerical evidence, we have a complete description of all finite volume form factors below the three-soliton threshold. This makes it possible to evaluate the one-point and two-point functions including all corrections below the three-soliton threshold using the methods developed in [2] and in [18], respectively. We must stress that while here we considered sine-Gordon theory, the results are expected to be valid for general integrable models, such as the  $O(3)$  nonlinear  $\sigma$  model, and are therefore potentially applicable to a range of field theoretic and condensed matter problems.



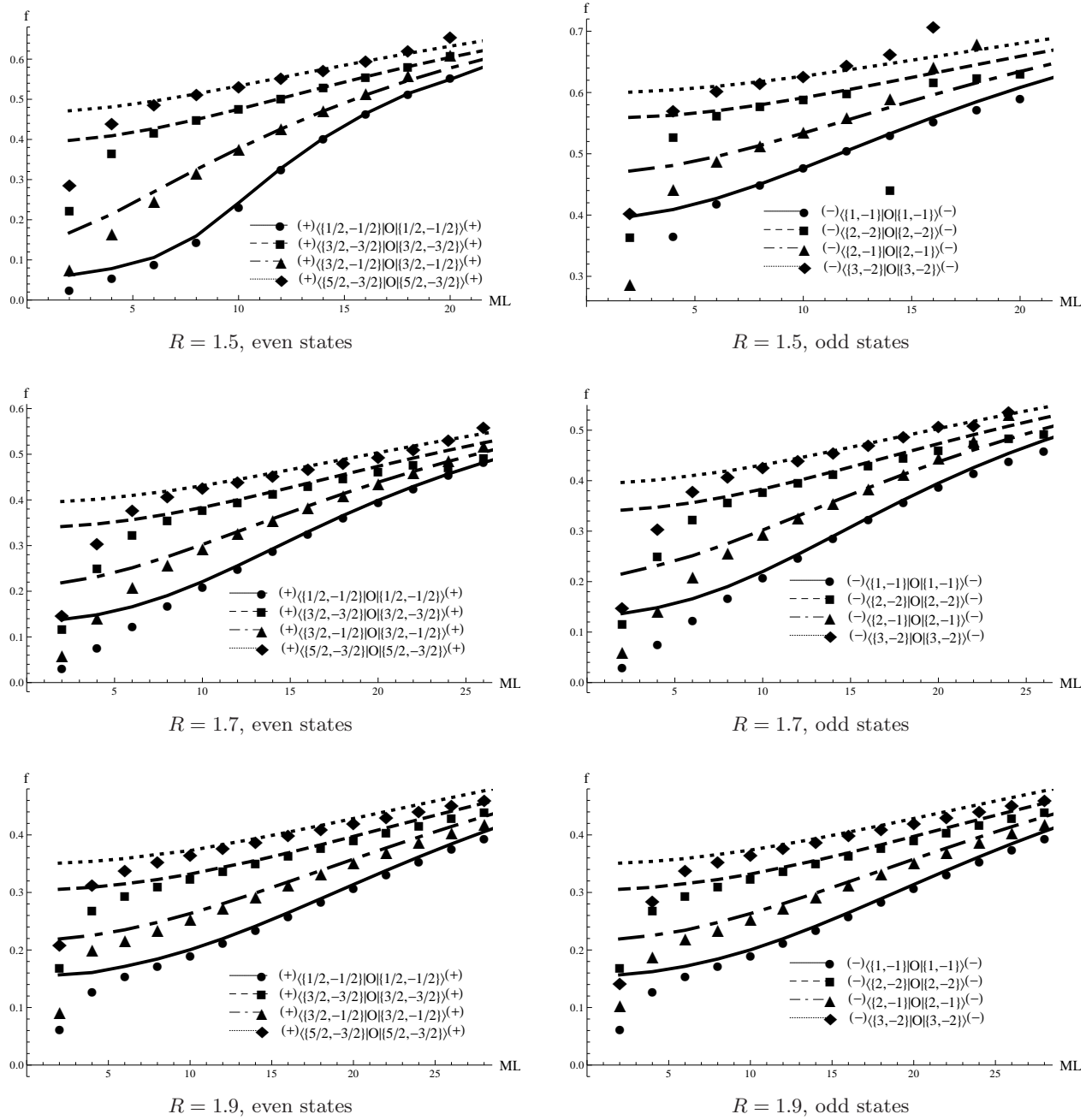


Figure 5.2: Diagonal matrix elements in even and odd soliton-antisoliton states

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